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(*A*, *B*)-Invariant Polyhedral Sets of Linear Discrete-Time Systems

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Abstract. The problem of confining the trajectory of a linear discrete-time system in a given polyhedral domain is addressed through the concept of (A, B)-invariance. First, an explicit characterization of (A, B)-invariance of convex polyhedra is proposed. Such characterization amounts to necessary and sufficient conditions in the form of linear matrix relations and presents two major advantages compared to the ones found in the literature: it applies to any convex polyhedron and does not require the computation of vertices. Such advantages are felt particularly in the computation of the supremal (A, B)-invariant set included in a given polyhedron, for which a numerical method is proposed. The problem of computing a control law which forces the system trajectories to evolve inside an (A, B)-invariant polyhedron is treated as well. Finally, the (A, B)-invariance relations are generalized to persistently disturbed systems.

Key Words. Linear systems, linear programming, convexity, polyhedra, invariance.

1. Introduction

Linear systems subject to pointwise-in-time constraints have proved to be objects of great interest for both theoreticians in optimization/control and practitioners. The usefulness of this model is due largely to the fact that, in real-life control problems, such constraints arise often from either physical limitations on input and output variables or the validity domain of linearization of nonlinear systems.

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In particular, the positive invariance approach has been used to solve a large number of problems on constrained dynamical systems. A set in the state space is positively invariant if any trajectory originated from this set does not leave it. It is a fact that physical limitations inherent to the operation of actual dynamical systems result very often in linear constraints on the state and/or control variables. As a consequence, much effort has been directed toward the development of the theory of positively invariant polyhedral sets, initially for discrete-time systems (Refs. 1–4), but also for continuous-time systems (Refs. 4–6). The most direct application of this theory to the solution of constrained control problems consists in verifying the existence of a state feedback control law which achieves the positive invariance of the polyhedron defined by the constraints. The drawbacks of this approach are twofold: closed-loop positive invariance of the polyhedron of constraints can be achieved seldom; the use of linear state feedback can restrict the possibility of achieving constraints satisfaction. Then, one is led to consider other sets, different from the set of constraints, and also more general control laws. Such considerations are naturally embedded in the concept of (A, B)-invariance.

In the framework of the geometric approach to the control of linear systems, the concept of (A, B)-invariance of subspaces plays an important role. In particular, it has been applied widely to the solution of some classical control problems, such as disturbance and input/output decoupling (Refs. 7–8). This concept of (A, B)-invariance, although with a different denomination, has been applied also to convex polyhedra, to characterize the possibility of controlling discrete-time systems subject to pointwise-in-time trajectory constraints.

In the sixties and seventies, seminal works on this subject (see e.g. Refs. 9–11) have treated this problem essentially at the conceptual level. Concerning more applied results, a vertex-by-vertex characterization has been proposed in Ref. 12 for compact polyhedra. In Ref. 13, this approach was used to solve a minimum-time control problem. Finally, in Ref. 14, the results of Ref. 12 were extended to uncertain additively disturbed systems. However, two major drawbacks can be detected in this approach. First, it applies only to compact polyhedra, whereas in many problems the polyhedron defined by state, control, or output constraints is not compact. Second, depending on the complexity of the polyhedron considered, the computation of its vertices and consequently the test of (A, B)-invariance can become expensive numerically.

In general, a polyhedron defined by linear constraints does not possess the (A, B)-invariance property. However, constraint satisfaction can be achieved if the initial states are forced to belong to an (A, B)-invariant set contained in the set of constraints. The set which is generally chosen, because it is the least conservative, is the supremal set or an approximation of it, that is, the set which contains all the other sets. Several algorithms have been proposed to compute this set (Refs. 14–15). As they are based on vertex-by-vertex characterization of (A, B)-invariance, these algorithms are demanding numerically. The basic contribution of this paper is in characterizing the property of (A, B)-invariance of convex polyhedra for discrete-time systems. By application of the Farkas lemma, necessary and sufficient conditions under which a general convex polyhedron is (A, B)-invariant are established in the form of linear matrix relations. A particular form of these relations is derived in the case of 0-symmetrical polyhedra. The problem of computing a control law which achieves closed-loop positive invariance of an (A, B)-invariant polyhedron is treated as well. A piecewise linear control law is proposed. It is an extension to the noncompact case of the control law proposed in Refs. 12, 14. $(A, B)\lambda$-contractive polyhedral sets are also introduced and characterized. Then, the supremal (A, B)-invariant sets contained in a given polyhedron are studied. Such sets are characterized theoretically and a numerical method, based on the (A, B)-invariance relations, is proposed for their computation.

Notation and Terminology. In mathematical expressions, a colon stands for "such that". By convention, inequalities between vectors and inequalities between matrices are componentwise. $\mathcal{N}$ and $\Re$ represent respectively the sets of natural and real numbers. I_n represents the identity matrix of order n. A vector or matrix is said to be nonnegative if all its components are nonnegative. The absolute value $|M|$ of a matrix M [resp. $|v|$ of a vector v] is defined as the matrix [resp. vector] of the absolute value of its components. M_i represents the ith row of M, and M_{ij} represents the element of row i and column j of matrix M. Let Ω be a set in a normed linear space $\mathscr{X}$, with the norm represented by $\|\cdot\|$. The set Ω is said to be bounded if there exists a scalar $s > 0$ such that $\|x\| \leq s$, $\forall x \in \Omega$; Ω is closed if it contains all its closure points; finally, Ω is compact if it is bounded and closed. In this work, only closed set are studied, and the linear spaces considered are over the field of real numbers $\Re$. The column vectors of a matrix M form a generating set of a polyhedral cone $\Re$ if and only if there exists a nonnegative vector ξ such that $x = M\xi$, $\forall x \in \mathscr{R}$. Each column vector of M is then called a generator of $\mathscr{R}$. A generating set is said to be a minimal generating set if it is defined by the smallest number of generators.

2. Characterization of (A, B)-Invariance

Consider the linear time-invariant discrete-time system described by

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

with $k \in \mathcal{N}$; $x \in \Re^n$ is the state vector and $u \in \Re^m$ is the control vector.

Definition 2.1. A nonempty closed set $\Omega \subset \Re^n$ is said to be positively invariant with respect to a dynamical system $x(k+1) = f(x(k))$ if

$$x(k) \in \Omega, \qquad \forall x(0) \in \Omega, \forall k \in \mathcal{N}.$$

Definition 2.2. A nonempty closed set $\Omega \subset \Re^n$ is said to be (A, B)-invariant with respect to the system (1) if

$$\exists \text{ a control vector } u \in \Re^m\text{: } Ax + Bu \in \Omega, \forall x \in \Omega.$$

In other words, Ω is (A, B)-invariant if, $\forall x(0) \in \Omega$, there exists a control sequence $\{u(k)\}$, $k \in \mathcal{N}$, such that the trajectory of the state vector of the controlled system is completely contained in Ω. The definition is analogous to that of (A, B)-invariant subspaces (Ref. 7) or controlled invariant subspaces (Ref. 8), but in a more general framework.

Definition 2.3. The one-step admissible set to Ω is defined as follows (Ref. 14):

$$\mathcal{Q}(\Omega) = \{x \in \Re^n\text{: } \exists u \in \Re^m\text{: } Ax + Bu \in \Omega\}.$$

By definition, $\mathcal{Q}(\Omega)$ is the set of all states which can be transferred to Ω in one step. Then, it is clear that the (A, B)-invariance of Ω is equivalent to the following geometric condition (Refs. 10–11).

Theorem 2.1. The set $\Omega \subset \Re^n$ is (A, B)-invariant with respect to the system (1) if and only if $\Omega \subset \mathcal{Q}(\Omega)$.

2.1. (A, B)-Invariance of Polyhedra: General Case. The study will now be restricted to convex polyhedra containing the origin, that is, to the case where

$$\Omega = R[G, \rho] = \{x\text{: } Gx \leq \rho\}, \qquad \rho \geq 0.$$

For a given time k, admissibility of the state vector at the time $k+1$ is characterized by the set of constraints

$$GAx(k) + GBu(k) \leq \rho. \tag{2}$$

These constraints define a convex polyhedron Π on the linear space defined by the extended vector $\left[\begin{smallmatrix} x(k) \\ u(k) \end{smallmatrix}\right]$. Then, the largest set of one-step admissible state vectors associated with (2) is the projection of Π onto the state space. An explicit expression of this projection can be obtained from the Farkas lemma; see e.g. Ref. 16.

Proposition 2.1. The one-step admissible set $\mathcal{Q}(R[G, \rho])$ is the convex polyhedron $R[TGA, T\rho]$, where the row vectors of the matrix T form a minimal generating set of the nonnegative left kernel of the matrix GB, defined by

$$\Gamma = \{w \in \Re^g: w \geq 0, (GB)^T w = 0\}. \tag{3}$$

Proof. The row vectors of T generates the polyhedral cone Γ. Then from the application of the Farkas lemma (Refs. 16–18),

$$\exists u: GBu \leq \rho - GAx \text{ iff } T(\rho - GAx) \geq 0. \qquad \square$$

As shown in Ref. 13, it is possible to compute the matrix T by means of the Fourier–Motzkin elimination technique (Ref. 16).

From Theorem 2.1 and Proposition 2.1, the (A, B)-invariance of $R[G, \rho]$ can be characterized geometrically by

$$R[G, \rho] \subset R[TGA, T\rho]. \tag{4}$$

This characterization can be translated into matrix relations by means of the following result, which can be derived from an extended version of the Farkas lemma, found in e.g. Ref. 3.

Theorem 2.2. The convex polyhedron $R[G, \rho] \subset \Re^n$ is (A, B)-invariant if and only if there exists a nonnegative matrix Y such that

$$YG = TGA, \tag{5}$$

$$Y\rho \leq T\rho. \tag{6}$$

One advantage of the above characterization is that Theorem 2.2 applies to any convex closed polyhedron, contrarily to the characterization proposed in Refs. 12, 14, which applies only to compact polyhedra. The second advantage is of a numerical nature. In the approach proposed in Refs. 12, 14 for testing (A, B)-invariance, one needs to compute first the vertices of the polyhedron, which is known to be a hard computational task for large-dimensional systems. Then, one has to test at each vertex for the existence of an admissible control. On the contrary, efficient methods are available to compute the matrix T; see e.g. Refs. 13, 17. Then, conditions (5)–(6) can be checked by means of the solution of a simple linear program.

The following additional properties can be pointed out:

(i) In the case $T = 0$, the polyhedron $R[TGA, T\rho]$ becomes the whole state space $\Re^n$, and condition (4) is satisfied trivially. Relations (5)–(6) are satisfied with $Y = 0$. The convex polyhedron $R[G, \rho]$ is then (A, B)-invariant trivially.

(ii) In the case of an autonomous system ($B=0$), Γ is the entire nonnegative orthant $\Re^g_+$, $T=I_g$, and relations (5)–(6) become the classical positive invariance relations; see e.g. Refs. 1–3.

(iii) Theorem 2.2 can be extended to the case when the control vector is subject to linear constraints,

$$u(k)\in\mathcal{U}=R[U,\psi]=\{u\in\Re^m: Uu\leq\psi\}, \qquad \forall k\in\mathcal{N}. \tag{7}$$

In this case, the convex polyhedron $R[G,\rho]$ is $\mathcal{U}$-(A,B)-invariant with respect to system (1) subject to the constraint $u(k)\in\mathcal{U}R[U,\psi]$ if and only if there exists a nonnegative matrix Y such that

$$YG=T_gGA, \tag{8}$$

$$Y\rho\leq T_g\rho+T_u\psi, \tag{9}$$

where the rows of the matrix $[T_g \;\; T_u]$ form a minimal generating set of the nonnegative left kernel of the matrix $[\begin{smallmatrix}GB\\U\end{smallmatrix}]$.

(iv) Finally, it is interesting to note from relations (5)–(6) that the (A,B)-invariance of the characteristic cone $R[G,0]=\{x: Gx=0\}$ is a necessary condition for the (A,B)-invariance of $R[G,\rho]$.

2.2. (A, B)-Invariance of Symmetrical Polyhedra. The case of 0-symmetrical polyhedra is now considered: $\Omega=S(Q,\mu)$, with

$$S(Q,\mu)=\{x: |Qx|\leq\mu\}.$$

Note that $S(Q,\mu)$ can be written in the form $R[G,\rho]$ with

$$G=\begin{bmatrix}Q\\-Q\end{bmatrix}, \qquad \rho=\begin{bmatrix}\mu\\\mu\end{bmatrix}.$$

Let $[T_1, T_2]$ be a matrix whose rows form a minimal set of generators of the polyhedral cone Γ; see (3), with G given above. Now, form the matrix $\mathcal{T}$ by deleting from the matrix T_1-T_2 the rows

$$T_{1i}-T_{2i} \text{ for which } T_{1i}-T_{2i}=0 \text{ or}$$

$$T_{1i}-T_{2i}=-T_{1j}+T_{2j} \text{ for some } j<i.$$

The following result specializes the (A,B)-invariance relations to the symmetrical case. The proof is similar to that of the continuous-time case, which can be found in Ref. 19.

Corollary 2.1. The symmetrical convex polyhedron $S(Q,\mu)\subset\Re^n$ is (A,B)-invariant with respect to the system (1) if and only if there exists a matrix Y such that

$$YQ = \mathscr{T}QA, \tag{10}$$

$$|Y|\mu \leq |\mathscr{T}|\mu. \tag{11}$$

Note that any row vector t belonging to the left kernel of the map QB can be written in the form

$$t = t_1 - t_2, \qquad t_1, t_2 \geq 0.$$

This means that $[t_1, t_2]^T$ belongs to Γ; see (3), with

$$G = \begin{bmatrix} Q \\ -Q \end{bmatrix}.$$

Therefore, the matrix $\mathscr{T}$ necessarily contains as a submatrix a row vector basis of the left kernel of map QB. Based on this fact, the following complementary result, which specializes Corollary 2.1 to the case of vector subspaces, can be proved easily.

Corollary 2.2. The subspace $\ker(Q) \in \mathfrak{R}^n$ is (A, B)-invariant if and only if there exists a matrix M such that $MQ = KQA$, where the row vectors of K span the left kernel of map QB.

Consider now the factor space $\bar{\mathscr{X}} = \mathfrak{R}^n/\ker(Q)$ and let $P: \mathfrak{R}^n \to \bar{\mathscr{X}}$ be the canonical projection. The following maps can be defined on $\bar{\mathscr{X}}$ (Ref, 7):

- (i) the map $\bar{A}$ induced in $\mathscr{X}$ by A, given by $\bar{A}P = PA$;
- (ii) the map $\bar{B}$, given by $\bar{B} = PB$;
- (iii) the map $\bar{Q}$, given by $\bar{Q}P = Q$.

Then, the following factor system can be defined in $\bar{\mathscr{X}}$ (Ref. 7):

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k), \tag{12}$$

as well as the polyhedron

$$\bar{S}(\bar{Q}, \mu) = \{\bar{x}; |\bar{Q}\bar{x}| \leq \mu\}, \tag{13}$$

and its extension to $\mathfrak{R}^n$

$$S(\bar{Q}, \mu) = \bar{S}(\bar{Q}, \mu)P. \tag{14}$$

Then, the polyhedron $S(Q, \mu)$ can be decomposed into the following form (Ref. 16):

$$S(Q, \mu) = \ker(Q) + S(\bar{Q}, \mu). \tag{15}$$

The following result shows that the test for (A, B)-invariance of unbounded symmetrical polyhedra can be decomposed.

Corollary 2.3. The symmetrical polyhedron $S(Q, \mu)$ is (A, B)-invariant with respect to the system (1) if and only if:

(i) the subspace $\ker(Q)$ is (A, B)-invariant;
(ii) the compact polyhedron $\bar{S}(\bar{Q}, \mu)$ is $(\bar{A}, \bar{B})$-invariant with respect to the system (12).

Proof. Initially, suppose that $S(Q, \mu)$ is (A, B)-invariant. The (A, B)-invariance of $\ker(Q)$ follows from (10), from the fact that the rows of $\mathscr{T}$ span the left kernel of QB, and from Corollary 2.2. Again from (10), there exists a matrix Y such that

$$Y\bar{Q}P = \mathscr{T}\bar{Q}PA = \mathscr{T}\bar{Q}\bar{A}P;$$

hence,

$$Y\bar{Q} = \mathscr{T}\bar{Q}\bar{A}.$$

This relation, together with (11), shows the $(\bar{A}, \bar{B})$-invariance of $S(\bar{Q}, \mu)$.

Conversely, suppose that $\ker(Q)$ is (A, B)-invariant and that $\bar{S}(\bar{Q}, \mu)$ is $(\bar{A}, \bar{B})$-invariant with respect to the system (12). From (15), every vector $x \in S(Q, \mu)$ can be written in the form

$$x = x^Q + x^S, \qquad \text{with } x^Q \in \ker(Q),\ X^S \in S(\bar{Q}, \mu).$$

By assumption, there exists a matrix F^Q such that $\ker(Q)$ is $(A + BF^Q)$-invariant and a control sequence

$$\{u^S(k)\},\ k \in \mathscr{N}, \text{ such that } x^S(k) \in \ker(Q) + S(\bar{Q}, \mu).$$

Then, it is clear that $S(Q, \mu)$ is positively invariant under the control law

$$u(k) = F^Q x^Q(k) + u^S(k). \qquad \square$$

2.3. $(A, B)\lambda$-Contractive Sets. In the case of compact (A, B)-invariant sets containing the origin, it is often important to increase the convergence rate of the trajectory to the equilibrium point. For example, this can help the system incorporate the effects of disturbances and/or uncertainties. (A, B)-invariance and convergence rate are conjugated in the following definition.

Definition 2.4. Given $0<\lambda\leq 1$, a compact set $\Omega\subset\Re^n$ is said to be (A, B)-invariant λ-contractive [or simply $(A, B)\lambda$-contractive] with respect to the system (1) if

there exists a control vector $u\in\Re^n$ such that $Ax+Bu\in\lambda\Omega$, $\forall x\in\Omega$.

It is clear that an (A, B)-invariant set is an $(A, B)\lambda$-contractive set with $\lambda=1$.

In the case where the origin belongs to the interior of a convex polyhedron $R[G, \rho]$, $\rho<0$, the condition (2) for one-step admissibility can be replaced by the $(A, B)\lambda$-contractivity condition,

$$GAx(k)+GBu(k)\leq\lambda\rho,$$

which results in the following points:

(i) Given a contraction rate λ, the one-step admissible set of the polyhedron $R[G, \rho]$ is the convex polyhedron $R[TGA, \lambda T\rho]$, where the matrix T is defined as in Theorem 2.2.

(ii) The convex polyhedron $R[G, \rho]\subset\Re^n$ is $(A, B)\lambda$-contractive with respect to the system (1) if and only if there exists a nonnegative matrix Y such that (5) is verified and

$$Y\rho\leq\lambda T\rho. \tag{16}$$

(iii) The convex symmetrical polyhedron $S(Q, \mu)\subset\Re^n$ is $(A, B)\lambda$-contractive with respect to the system (1) if and only if there exists a nonnegative matrix Y such that (10) is verified and

$$|Y|\mu\leq\lambda|\mathscr{T}|\mu. \tag{17}$$

3. Supremal (A, B)-Invariant Set

Suppose that the state of the system (1) is subject to the constraint $x\in\Omega$. In general, the set Ω is not (A, B)-invariant. However, a possible solution to the constrained problem is to restrict the state to an (A, B)-invariant set contained in Ω. It is also desirable that, in some sense, this set be as large as possible. To refine this issue, consider the following property, whose proof is straightforward.

Proposition 3.1. The family of all (A, B)-invariant sets contained in a convex set Ω is closed under the operation "convex hull of the union".

Since Ω is closed by assumption, this proposition guarantees the existence [in the family of (A, B)-invariant sets contained in Ω] of a supremal element (an element which contains all the other elements),

$$\mathscr{C}^{\infty}(\Omega) \triangleq \text{supremal } (A, B)\text{-invariant set contained in } \Omega.$$

Indeed, $\mathscr{C}^{\infty}(\Omega)$ is the set defined by the convex hull of the union of all (A, B)-invariant sets in Ω. The supremal set can be characterized by the following recurrence formula (Ref. 14):

$$\mathscr{C}^{i+1} = \mathscr{Q}(\mathscr{C}^{i}) \cap \mathscr{C}^{i}, \qquad \text{with } \mathscr{C}^{0} = \Omega, \tag{18}$$

$$\mathscr{C}^{\infty}(\Omega) = \lim_{i \to \infty} \mathscr{C}^{i}. \tag{19}$$

It should be noticed that the set $\mathscr{C}^{i}$ is the set of states for which there exists a control sequence able to force them to stay in Ω in i steps. According to (18), $\mathscr{C}^{i} \subset \mathscr{C}^{i-1}$, and the supremal set is obtained for $i \to \infty$.

One can introduce also a contraction rate λ and adapt the recurrence (18)–(19) to compute the set

$$\mathscr{C}^{\infty}(\Omega, \lambda) \triangleq \text{supremal } (A, B)\lambda\text{-contractive set contained in } \Omega.$$

For doing so, it suffices to replace the recurrence (18)–(19) by

$$\mathscr{C}^{i+1} = \mathscr{Q}(\lambda\mathscr{C}^{i}) \cap \mathscr{C}^{i}, \qquad \text{with } \mathscr{C}^{0} = \Omega, \tag{20}$$

$$\mathscr{C}^{\infty}(\Omega, \lambda) = \lim_{i \to \infty} \mathscr{C}^{i}. \tag{21}$$

The supremal (A, B)-invariant set contained in $R[G, \rho]$ can be constructed by means of the recurrence (18)–(19). Such a construction requires generally a large computational effort. Along the iterative process, many redundant inequalities may be generated. Therefore, it is particularly desirable to implement an algorithm which generates only nonredundant inequalities at each iteration. This property can be achieved using the (A, B)-invariance relations (5)–(6), as shown in the following algorithm (Ref. 20).

Step 1. Initialize $i=1$, $l^0=0$, $T^0=0$, $t^0=0$, $G^0=G$, $\rho^0=\rho$, $g^0=g$. Define a precision ϵ.

Step 2. Compute the matrix $T^i \in \mathfrak{R}^{t^i \times g^i}$, whose rows form a generating set of the polyhedral cone

$$\Gamma^i = \{w;\ (G^iB)^T w = 0,\ w \geq 0\},$$

and decompose T^i in the form

$$T_i = \begin{bmatrix} T^{i-1} & \vdots & 0 \\ \hdashline & U_i & \end{bmatrix}, \qquad U^i \in \mathfrak{R}^{r^i \times g^i}. \tag{22}$$

Step. 3. Solve by linear programming the following problems, for $j=1,\ldots,r^i$:

$$\min_{Y_j^i} \quad Y_j^i\rho^i, \tag{23a}$$

$$\text{s.t.} \quad Y_j^iG^i = U_j^iG^iA, \tag{23b}$$

$$Y_j^i \geq 0. \tag{23c}$$

If $Y_j^i\rho^i - U_j^i\rho^i \leq \epsilon$, $\forall j$, then $\mathscr{C}^\infty(R[G,\rho]) = R[G^i,\rho^i]$; stop. If $\exists j$: $Y_j^i\rho^i - U_j^i\rho^i > \epsilon$, order the rows of U^i and Y^i by means of a permutation matrix P^i in the form

$$P^iU^i = \begin{bmatrix} U_1^i \\ U_2^i \end{bmatrix}, \qquad P^iY^i = \begin{bmatrix} Y_1^i \\ Y_2^i \end{bmatrix},$$

with $Y_2^i \in \Re^{l^i \times r^i}$, so that

$$Y_1^i\rho^i - U_1^i\rho^i \leq \epsilon\mathbb{1}, \tag{24}$$

$$Y_2^i\rho^i - U_2^i\rho^i > \epsilon\mathbb{1}. \tag{25}$$

Step 4. Construct the matrices

$$G^{i+1} = \begin{bmatrix} G^i \\ U_2^iG^iA \end{bmatrix}, \qquad \rho^{i+1} = \begin{bmatrix} \rho^i \\ U_2^i\rho^i \end{bmatrix}.$$

Do $g^{i+1} = g^i + l^i$.

Step 5. Do $i = i+1$, and return to Step 2.

The fact that this algorithm converges to $\mathscr{C}^\infty(R[G,\rho])$ follows from the next result.

Proposition 3.2. The polyhedron $R[G^{i+1},\rho^{i+1}]$ is identical to the set $\mathscr{Q}(R[G^i,\rho^i]) \cap R[G^i,\rho^i]$.

Proof. From Proposition 2.1 and (22), the set $\mathscr{Q}(R[G^i,\rho^i]) \cap R[G^i,\rho^i]$ is defined by the inequalities

$$G^ix \leq \rho^i,$$

$$U_1^iG^iAx \leq U_1^i\rho^i,$$

$$U_2^iG^iAx \leq U_2^i\rho^i.$$

Note that the rows associated to T^{i-1} in (22) have been considered already in the computation of $R[G^i,\rho^i]$.

The inclusion of $\mathcal{Q}(R[G^i, \rho^i]) \cap R[G^i, \rho^i]$ in $R[G^{i+1}, R^{i+i}]$ is evident. Conversely, every point $x \in R[G^{i+i}, \rho^{i+1}]$ verifies, with the nonnegative matrix Y_1^i verifying (23)–(24),

$$U_1^i G^i A x = Y_1^i G^i x \leq Y_1^i \rho^i \leq U_1^i \rho^i + \epsilon \mathbb{1},$$

which implies,up to the given precision,

$$R[G^{i+i}, \rho^{i+1}] \subset \mathcal{Q}(R[G^i, \rho^i]) \cap R[G^i, \rho^i]. \qquad \square$$

Concerning this algorithm, the following should be pointed out.

(i) For some cases [namely, when rank $(G^i) < n$], the linear program (23) may not be solvable for some values of j. In this case, the associated rows U_j^i and Y_j^i must be included in matrices U_2^i and Y_2^i, respectively; see (25).

(ii) The set $\mathscr{C}^\infty(\Omega)$ is polyhedral if and only if it is generated in a finite number of iterations. However, it can be approximated by a polyhedron. Indeed, it has been shown in Ref. 14 that, given the set $\mathscr{C}^\infty(\Omega, \lambda)$, $0 < \lambda \leq 1$, then $\forall \lambda'$ such that $\lambda \leq \lambda' \leq 1$, $\exists i'$ such that $\mathscr{C}^{i'}$ is λ'-contractive, with $\mathscr{C}^{i'}$ given by the recurrence (20). Therefore, an approximation of $\mathscr{C}^\infty(\Omega)$ can be computed through the recurrence (20), with λ close to 1, until eventually obtaining an (A, B)-invariant polyhedron $\mathscr{C}^i$.

It is interesting to note that, for unbounded symmetrical polyhedra, the computation of $\mathscr{C}^\infty(\Omega)$ can be performed in a decomposed manner. Then, let $S(Q, \mu)$ be an unbounded symmetrical polyhedron, and let $\mathscr{V}^*$ be the largest (A, B)-invariant subspace contained in $\ker(Q)$. Consider also the factor space $\hat{\mathscr{X}} = \Re^n / \mathscr{V}^*$ and the reduced-order system

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k). \tag{26}$$

The maps $\hat{A}$ and $\hat{B}$ are defined from the orthogonal projection $P_v: \Re^n \to \hat{\mathscr{X}}$, the same way as their analogs in the system (12). The polyhedron $S(Q, \mu)$ is now decomposed in the following form:

$$S(Q, \mu) = \mathscr{V}^* + S(\hat{Q}, \mu),$$

where

$$S(\hat{Q}, \mu) = \hat{S}(\hat{Q}, \mu) P_v, \qquad \text{with } \hat{S}(\hat{Q}, \mu) \triangleq \{\hat{x};\, |\hat{Q}\hat{x}| \leq \mu\}.$$

Define also the set

$$\hat{\mathscr{C}}^\infty(\hat{S}(\hat{Q}, \mu)) \triangleq \text{supremal } (\hat{A}, \hat{B})\text{-invariant set contained in } \hat{S}(\hat{Q}, \mu),$$

and its extension to $\Re^n$,

$$\hat{\mathscr{C}}^\infty(S(Q,\mu)) = \hat{\mathscr{C}}^\infty(\hat{S}(\hat{Q},\mu))P_v.$$

Theorem 3.1. The supremal (A, B)-invariant set contained in the polyhedron $S(Q, \mu)$ can be decomposed as follows:

$$\mathscr{C}^\infty(S(Q,\mu)) = \mathscr{V}^* + \hat{\mathscr{C}}^\infty(S(Q,\mu)).$$

Proof. For all $x \in \mathscr{V}^* + \hat{\mathscr{C}}^\infty(S(Q,\mu))$, there exist $x^\omega \in \mathscr{V}^*$ and $x^C \in \hat{\mathscr{C}}^\infty(S(Q,\mu))$ such that $x = x^v + x^C$. From the construction of $\mathscr{V}^*$ and $\hat{\mathscr{C}}^\infty(S(Q,\mu))$, $\exists F^v$ such that $\mathscr{V}^*$ is $(A + BF^v)$-invariant and there exists a control sequence $\{u(k)\}$, $k \in \mathscr{N}$, such that $x^C(k) \in \mathscr{V}^* + \hat{\mathscr{C}}^\infty(S(Q,\mu))$. Hence, $\mathscr{V}^* + \hat{\mathscr{C}}^\infty(S(Q,\mu))$ is an (A, B)-invariant set contained in $S(Q, \mu)$.

Conversely, consider a generic point $x(k) \in \mathscr{C}^\infty(S(Q,\mu))$. Since $\mathscr{C}^\infty(S(Q,\mu))$ is (A, B)-invariant, then there exists a control $u(k)$ such that $x(k+1) \in S(Q,\mu)$. The vector $x(k) \in S(Q,\mu)$ can be decomposed in the following form:

$$x(k) = x^v(k) + x^s(k), \qquad \text{with } x^v(k) \in \mathscr{V}^* \text{ and } x^s(k) \in S(\hat{Q},\mu).$$

The control vector $u(k)$ can be decomposed in the form

$$u(k) = F^v x^v(k) + u^s(k), \qquad \text{with } \mathscr{V}^* \ (A + BF^v)\text{-invariant}.$$

Therefore,

$$x^v(k+1) = (A + BF^v)x^v(k) \in \mathscr{V}^*.$$

Consider now the system (26), and suppose that $x^s(k) \notin \hat{\mathscr{C}}^\infty(S(Q,\mu))$. One can then verify that

$$\hat{x}^s(k+1) = \hat{A}\hat{x}^s(k) + \hat{B}u^s(k) \notin \hat{S}(\hat{Q},\mu),$$

which contradicts the assumption $x(k) \in \mathscr{C}^\infty(S(Q,\mu))$. Hence,

$$\mathscr{C}^\infty(S(Q,\mu)) \subset \mathscr{V}^* + \hat{\mathscr{C}}^\infty(S(Q,\mu)),$$

and the proof is complete. □

4. Control Law Computation

The satisfaction of the (A, B)-invariance conditions guarantees, for any point in the polyhedron considered, the existence of a control which forces the state trajectory to stay in it. However, that does not presuppose a particular type of control law. Nevertheless, it is important, even mandatory in practice, that the system be controlled by means of a closed-loop control law.

An answer to this question has been given in Ref. 12 and has been improved in Ref. 14: a piecewise linear control law was proposed to achieve the closed-loop positive invariance of compact (A, B)-invariant polyhedra. To this end, the polyhedral sets are divided in regions defined by the convex hull of the origin and n vertices, where n is the dimension of the system. Then, a state feedback control law is computed for each region. The control law described in the sequel extends such a law to the case of noncompact polyhedra.

4.1. General Case. Any polyhedron $R[G, \rho]$ can be decomposed in the form of the sum of the characteristic cone $R[G, 0]=\{x: Gx=0\}$ and a polytope Π (Ref. 16). Then, a set of admissible controls $(v_1, \ldots, v_p)$ can be associated to the vertices $(x_1, \ldots, x_p)$ of Π. They verify

$$GAx_i + GBv_i \leq \rho.$$

Similarly, a set of controls $(w_1, \ldots, w_q)$ can be associated to the set of generators M_j of $R[G, 0]$, so that

$$GAM_j + GBw_j \leq \rho.$$

Each point in $R[G, \rho]$ is represented by the set of coordinates $(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p)$ through the relation

$$x=\sum_{j=1}^{q} \alpha_j M_j + \sum_{i=1}^{p} \beta_i x_i, \qquad \text{with } \alpha_j \geq 0, \forall j, 0 \leq \beta_i \leq 1, \forall i, \sum_{i=1}^{p} \beta_i \leq 1. \tag{27}$$

Then, the following control function can be considered:

$$u(x)=\sum_{j=1}^{q} \alpha_j w_j + \sum_{i=1}^{p} \beta_i v_i. \tag{28}$$

For $\rho \geq 0$, a partition of $R[G, \rho]$ can be derived from the parametrization (27). Each region $\mathscr{X}_r$ of $R[G, \rho]$ is generated from the relation (27) by a set of generators and/or vertices $(M_{j'}, x_{i'})$, $j' \in \mathscr{J}_r$, $i' \in \mathscr{I}_r$, such that:

(i) card $\mathscr{J}_r) + \text{card}(\mathscr{I}_r) = n$, where card(·) represents the number of elements in the set considered.

(ii) A point $x \in \mathscr{X}_r$ is given by

$$x=\sum_{j' \in \mathscr{J}_r} \alpha_{j'} M_{j'} + \sum_{i' \in \mathscr{I}_r} \beta_{i'} x_{i'}, \tag{29a}$$

$$\text{with } \alpha_{j'} \geq 0, \qquad 0 \leq \beta_{i'} \leq 1, \qquad \sum_{i' \in \mathscr{I}_r} \beta_{i'} \leq 1. \tag{29b}$$

The transition between two adjacent regions is characterized by a pivoting operation for which one of the coefficients $(\alpha_{\bar{j}}, \beta_{\bar{i}})$ vanishes and either a generator M_j, $j \notin \mathscr{J}_r$, or a vertex x_i, $i \notin \mathscr{I}_r$, replaces [in the representation (29)] the generator or vertex for which either $\alpha_{\bar{j}}$ or $\beta_{\bar{i}}$ has vanished. The interior of the intersection of two adjacent regions is empty, and the union of all regions is the polyhedron $R[G, \rho]$.

Let X_r be a square matrix whose columns are the generators/vertices which define the region $\mathscr{X}_r$, and let U_r be a matrix whose columns are the associated control vectors w_j, v_i. A piecewise linear control law is then given by

$$u(k) = F_r x(k) = U_r(X_r)^{-1} x(k), \qquad x(k) \in \mathscr{X}_r. \tag{30}$$

This law is a possible realization of the law (28). Since compact polyhedra are defined completely by their vertices, the regions in which they are divided are compact polyhedra formed by the convex hull of the origin and n vertices. In this case, the control law (30) becomes that proposed in Refs. 12, 14.

4.2. Symmetrical Polyhedra. For unbounded symmetrical polyhedra, the control law can be computed in a decomposed manner. Consider the unbounded symmetrical polyhedron $S(Q, \mu)$ and the matrix P defined (as in Section 3.2) as the canonical projection of $\mathfrak{R}^n$ onto the factor space $\bar{\mathscr{X}} = \mathfrak{R}^n/\ker(Q)$.

The following result can be established.

Corollary 4.1. If the polyhedron $S(Q, \mu)$ is (A, B)-invariant with respect to the system (1), then a control law such that it is positively invariant in closed loop is given by

$$u(k) = F^Q x(k) + u^S(k),$$

where F^Q is such that $\ker(Q)$ is $(A + BF^Q)$-invariant and $u^S(k)$ is a control law such that the compact polyhedron $\bar{S}(\bar{Q}, \mu)$ [see (13)] is positively invariant with respect to the system

$$\bar{x}(k+1) = (\bar{A} + \bar{B}\bar{F}^Q)\bar{x}(k) + \bar{B}u(k),$$

where $\bar{F}^Q$ is defined by $\bar{F}^Q P = F^Q$.

Proof. Suppose that $x(k) \in S(Q, \mu)$. Under the proposed control law, one has

$$x(k+1) = (A + BF^Q)x(k) + Bu^S(k).$$

Hence,

$$\begin{aligned} Px(k+1) &= \bar{x}(k+1) \\ &= P(A+BF^Q)x(k) + PBu^S(k) \\ &= (\bar{A}+\bar{B}\bar{F}^Q)\bar{x}(k) + \bar{B}u^S(k). \end{aligned}$$

Then by assumption, $u^S(k)$ is such that $\bar{x}(k+1) \in \bar{S}(\bar{Q}, \mu)$. The existence of $u^S(k)$ follows from Corollary 2.3 and from the fact that, if $\bar{S}(\bar{Q}, \mu)$ is $(\bar{A}, \bar{B})$-invariant, then it is $(\bar{A}+\bar{B}\bar{F}^Q, \bar{B})$-invariant as well. In addition, since $\bar{x}(k+1) \in \bar{S}(\bar{Q}, \mu)$, then

$$x(k+1) \in \ker(Q) + \bar{S}(\bar{Q}, \mu) = S(Q, \mu). \qquad \square$$

5. Persistently Disturbed Systems

Consider the following linear discrete-time system:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k), \tag{31}$$

where $d \in \Re^q$ is a disturbance vector, supposed constrained to evolve inside a bounded domain $\mathscr{D} \subset \Re^q$,

$$d(k) \in \mathscr{D}, \qquad \forall k \in \mathscr{N}. \tag{32}$$

One can notice that this kind of disturbance acts persistently in time, and its energy is infinite. This is why it is named persistent disturbance by some authors.

Definition 5.1. A nonempty closed set $\Omega \subset \Re^n$ is said to be $\mathscr{D}$-(A, B)-invariant with respect to the system (31)–(32) if

$$\exists \text{ a control vector } u \text{ such that } Ax + Bu + Ed \in \Omega, \ \forall x \in \Omega, \ \forall d \in \mathscr{D}.$$

This definition assumes that the disturbance vector is not measured. The case of measurable disturbances has been considered in Ref. 18.

The one-step admissible set is now defined as in Ref. 14,

$$\mathscr{Q}(\Omega, \mathscr{D}) = \{x \in \Re^n : \exists u \in \Re^m : Ax + Bu + Ed \in \Omega, \ \forall d \in \mathscr{D}\}. \tag{33}$$

Therefore, the set Ω is $\mathscr{D}$-(A, B)-invariant with respect to (31)–(32) if and only if $\Omega \subset \mathscr{Q}(\Omega, \mathscr{D})$.

Consider now the polyhedral case,

$$\Omega = R[G, \rho], \qquad \mathscr{D} = R[D, \omega],$$

with

$$R[D, \omega] = \{d \in \Re^q : Dd \leq \omega\}.$$

Define the components δ_i of the vector δ as follows:

$$\delta_i = \max_{d \in R[D, \omega]} G_i E d.$$

For a given k, admissibility of the state vector in $k+1$ is now characterized by

$$GAx(k) + GBu(k) \leq \rho - \delta. \tag{34}$$

One can notice that the role of the vector δ is to absorb the effect of the disturbances. Then, the following results can be derived.

Proposition 5.1. In the polyhedral case, the one-step admissible set is given by

$$\mathscr{Q}(R[G, \rho], R[D, \omega]) = R[TGA, T(\rho - \delta)],$$

where the rows of the matrix T form a minimal generating set of the nonnegative left kernel of the matrix GB.

Theorem 5.1. The convex polyhedron $R[G, \rho]$ is $\mathscr{D}$-(A, B)-invariant with respect to the system (31)–(32), with $\mathscr{D} = R[D, \omega]$, if and only if there exists a nonnegative matrix Y such that

$$YG = TGA, \tag{35}$$

$$Y\rho \leq T(\rho - \delta). \tag{36}$$

As for the undisturbed case, one can show the existence of a supremal $\mathscr{D}$-(A, B)-invariant set contained in a given set Ω,

$$\mathscr{C}^\infty(\Omega, \mathscr{D}) \triangleq \text{supremal } \mathscr{D}\text{-}(A, B)\text{-invariant set contained in } \Omega,$$

which is given by

$$\mathscr{C}_{i+1} = \mathscr{Q}(\mathscr{C}_i, \mathscr{D}) \cap \mathscr{C}_i, \qquad \text{with } \mathscr{C}_0 = \Omega,$$

$$\mathscr{C}^\infty(\Omega, \mathscr{D}) = \lim_{i \to \infty} \mathscr{C}_i.$$

An algorithm for computing $\mathscr{C}^\infty(\Omega, \mathscr{D})$, with $\Omega = R[G, \rho]$, $\mathscr{D} = R[D, \omega]$, can be derived easily from the algorithm of Section 3.

A practical application of $\mathcal{D}$-(A, B)-invariant polyhedra is in the solution of the persistent disturbance attenuation problem, also known as the l^1-control problem; see e.g. Refs. 21-22. In particular, it has been shown that the possibility of achievement of a given l^1-performance level can be characterized by the existence of a nonempty internally stabilizable $\mathcal{D}$-(A, B)-invariant domain contained in the polyhedral region of the performance constraints.

6. Numerical Example

Consider the system (1) for which

$$A=\begin{bmatrix}0.4 & 0.9\\ 0.6 & 1.8\end{bmatrix}, \qquad B=\begin{bmatrix}0\\ 1\end{bmatrix},$$

and the polyhedron $R[G, \rho]$ with

$$G=\begin{bmatrix}0.2 & 0.2\\ -1 & -1\\ -1 & 0.35\\ 0.25 & -0.5\end{bmatrix}, \qquad \rho=\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix}.$$

The computation of the largest $(A, B)\lambda$-contractive set contained in $R[G, \rho]$, with $\lambda=0.8$, results in $\mathscr{C}(R[G, \rho], \lambda)=R[G^1, \rho^1]$, with

$$G^1=\begin{bmatrix} G \\ \hdashline 0.6 & 1.35\\ -1.5429 & -3.4714\end{bmatrix}, \qquad \rho^1=\begin{bmatrix}\rho\\ \hdashline 5.6\\ 3.0857\end{bmatrix}.$$

For $R[G^1, \rho^1]$, a matrix T^1 whose rows generate the nonnegative left kernel of G^1B, and a matrix Y^1 which verifies the conditions of

$(A, B)\lambda$-contractivity (5), (16) are given by

$$T^1 = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0.2881 \\ 0 & 1 & 2.8571 & 0 & 0 & 0 \\ 0 & 0 & 2.8571 & 2 & 0 & 0 \\ 0 & 0 & 2.8571 & 0 & 0 & 0.2881 \\ 0 & 1 & 0 & 0 & 0.7407 & 0 \\ 0 & 0 & 0 & 2 & 0.7407 & 0 \\ 0 & 0 & 0 & 0 & 0.7407 & 0.2881 \end{bmatrix},$$

$$Y^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.3889 \\ 0 & 0 & 0 & 0 & 0 & -0.1440 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0.6111 \\ 0 & 0 & 0 & 0 & 0 & 0.8560 \\ 0 & 0 & 0 & 0 & 0 & 0.1440 \\ 0 & 0 & 0 & 0 & 0 & -0.2449 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The polyhedral sets $R[G, \rho]$ and $R[G^1, \rho^1]$ are displayed on Fig. 1.

Figure 2 presents the set $R[G^1, \rho^1]$ divided in 6 regions, as described in Section 4.1. A trajectory of the state starting from one of the vertices, obtained through the application of a control law of the same type as (30), is represented by a dotted line in Fig. 2.

7. Conclusions

This work has studied the concept of (A, B)-invariance applied to polyhedral sets of the state space of linear systems. This concept has proven to be of fundamental importance to the control of constrained systems. An explicit characterization of (A, B)-invariance for discrete-time systems has been proposed, which amounts to necessary and sufficient conditions in the form of linear matrix relations. The advantages of such a characterization, when compared to the ones found in the literature, are twofold: it applies to any convex polyhedron and it does not demand the computation of

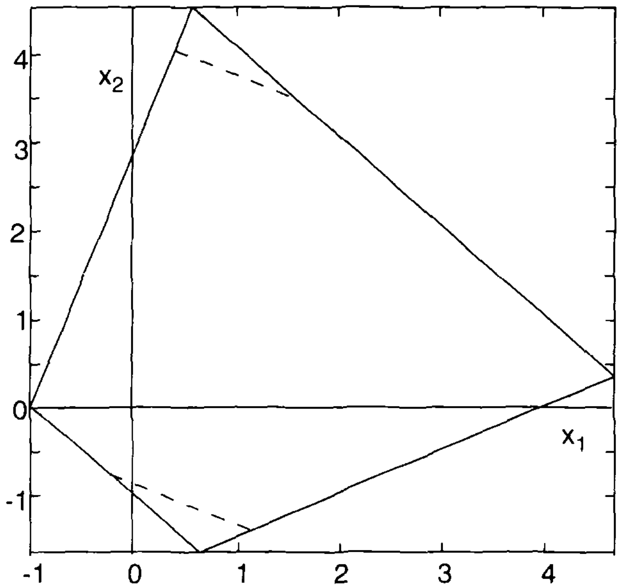

Fig. 1. Sets $R[G, \rho]$ and $\mathscr{C}^{\alpha}(R[G, \rho], \lambda)$.

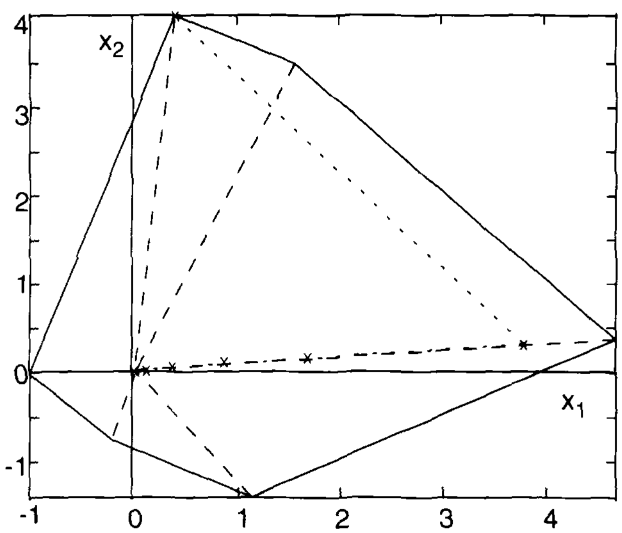

Fig. 2. Polyhedron $\mathscr{C}^{\alpha}(R[G, \rho], \lambda)$ divided in regions.

vertices. These advantages are particularly felt in computing the supremal (A, B)-invariant set contained in a given polyhedron. A numerical method has also been proposed for this computation, which uses the (A, B)-invariance relations to generate only nonredundant inequalities and to furnish an efficient test for convergence.

All these results have been specialized to the case of 0-symmetrical polyhedral sets. In particular, it has been shown that (A, B)-invariance of an unbounded symmetrical polyhedron is equivalent to the (A, B)-invariance of a subspace plus the (A, B)-invariance of a compact polyhedron associated to a reduced-order system.

The problem of computing a control law which achieves closed-loop positive invariance of an (A, B)-invariant polyhedron has also been considered. A piecewise linear state feedback control law, proposed in the literature for compact polyhedra, has been extended to the general case. However, the implementation of this law is very complex. We think that the search for simpler control laws should continue.

It has also been shown how the (A, B)-invariance results can be extended to systems subject to bounded additive disturbances. An extension of this work to continuous-time systems can be found also in Ref. 19.

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